Uncapacitated lot-sizing problem with production time windows, early production, backlog and lost sale

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Abstract

We consider the single item uncapacitated lot-sizing problem with production time windows, lost sale, early production and backlog over a planning horizon of T periods. In this context, a demand not processed within its time window can either be lost (lost sale), satisfied from a production that is processed before the release period of the demand (early production) or satisfied from a production that occurs after the demand due period (backlog). We present several properties of the optimal solution for different variants of the problem when production time windows are non customer specific. We propose dynamic programming algorithms to solve optimally the examined problems in $O(T^2)$.

Keywords: Lot-sizing, dynamic programming, production time windows, lost sale, early production, backlog.

1 Problem context and definition

In this paper, we address the single-level single-item uncapacitated lot-sizing problem with time window constraints and quality of service criteria. It is a production planning problem where a time varying demand has to be produced over a planning horizon of length T. In this work, we focus on production time windows, i.e. intervals where production can occur. In this context, the production of an item takes place between a release date that is the first period in which the item can be produced and a due date which corresponds to the latest customer order delivery date of the item. The release dates represent several situations such as the availability dates imposed by perishable products or the availability of raw materials or some semi-finished products involved in the production of the item. This is particularly important in a multi-level environment.

Two kinds of time window constraints can be considered, the delivery time windows [9] and the production time windows [7]. The delivery time window, also

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called grace period, represents the interval where the products can be delivered to the customers with no penalty. However, all the demands can still be produced as early as the first period of the planning horizon and the holding cost is considered to be null if a demand is satisfied within its corresponding time window. Dauzère-Pérès et al. [7] were the first to study the problem where a demand is represented by a time window. They define the general case of the problem where no restriction on the time window structure is imposed. Such a problem is called Customer Specific (CS). They use a pseudo-polynomial dynamic programming algorithm to solve the problem. More recently, Hwang [8] proposed a dynamic programming algorithm to solve the problem with concave production costs in $O(T^5)$.

An interesting special case of the problem is called Non Customer Specific (NCS) where two time windows cannot be strictly included. In other words, for any pair of time windows $[s_1, t_1]$, $[s_2, t_2]$, either $s_1 \leq s_2$ and $t_1 \leq t_2$ or $s_1 \geq s_2$ and $t_1 \geq t_2$. The NCS problem is widely encountered in practical situations. Moreover, it is solved in polynomial time. An $O(T^4)$ time dynamic programming algorithm was proposed in [7] and later the complexity was improved in $O(T^2)$ by Wolsey [14].

The problem considered in this paper includes quality of service criteria which aim at meeting the customer requirements as best as possible. In classical lot-sizing models, when the demand cannot be produced within the time window because of a lack of resource capacity or prohibitive production costs, then it is lost. In this paper, the possibility of producing the demand before the release period (early production) or after the due period (backlog) at a given cost is studied. The motivation comes from the attempt to satisfy the customer even if a penalty is induced, such additional cost will be preferred to the lost sale. More precisely, lost sales involve allowing some orders to not being delivered if the total cost of producing the demand is prohibitive. This situation illustrates a crucial lack in the production resource capacity for instance. Sandbothe and Thompson [11] are among the first who studied the uncapacitated lot-sizing problem including lost sales. They propose a dynamic programming algorithm to solve it in $O(T^3)$. Aksen et al. [4] reduce this complexity to $O(T^2)$ and Absi and Kedad-Sidhoum [1, 2] consider lost sales when several orders and setup times are considered. We can also quote the more recent work of van den Heuvel and Wagelmans [12] who show that there exists an equivalence between the lot-sizing problems dealing with bounded inventory, remanufacturing option, production time windows and cumulative capacities. Another criterion studied for the first time in this paper consists in considering early production where orders are produced before their release periods at a given penalty. Symmetrically, backlogs allow producing orders after their due periods at a given cost. For some customers, it is preferable to receive orders after the due period rather than not receive them at all. We quote the seminal work on backlogs of Zangwill [15].

The different parameters of the problem are defined as follows. Producing one unit in period t incurs a production cost p_t. The holding cost of one unit at the end of period t is h_t. A setup cost s_t is incurred at each period where production

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takes place. The demands are deterministic but have a particular structure. A demand becomes available for production at a given period r, its release period, before which it cannot be produced. A demand has to be produced before a due period t, $r \leq t$. This demand is represented by d_{rt}. Also, D_t defines the aggregate demand with due period at t, i.e. $D_t = \sum_{r=1}^{t} d_{rt}$. When modeling the fact that d_{rt} is not satisfied within its time window, three cases will be considered: Lost sale, early production and backlog. In the first case, demands that are not delivered at the end of a given period are lost and thus cannot be satisfied later, a unitary cost l_t is induced. When considering time windows, it is possible to produce the demands before their release periods at a given unit cost denoted e_t. Symmetrically, when the production of a demand starts after its due period, a backlog can occur at unit cost b_t. We will assume, without loss of generality, that $l_t > e_t > p_t$. The problem consists in finding a production plan over T that minimizes the total cost including production, setup and holding costs as well as the early production, backlog and lost sale costs. All the results presented in this paper concern the case of non customer specific demands.

The rest of the paper is organized as follows: Section 2 describes our models and clarify the integration of early production, lost sale and backlog to the classical lot-sizing problem with time windows. Mixed Integer Programming (MIP) formulations of these problems are presented as well as some numerical examples. Section 3 presents several properties and lemmas for the studied problems. Based on these properties, dynamic programming algorithms to solve the problems are presented in Section 4. The execution of the algorithms on the examples of Section 2 are detailed. Finally, Section 5 provides a short conclusion and future research directions.

2 Mathematical formulations

When considering quality service criteria, two core problems are to be studied. The first one allows lost sale and early production. In practical applications, this problem illustrates the situation where, if the cost of producing the demand within its time window is prohibitive, then we try to anticipate the production at a given cost. This can represent for instance the cost of early delivering raw materials to process the demand. The release period of the demand is violated and a cost is incurred. If the early production cost is still prohibitive then the demand is lost. The first problem related to these assumptions will integrate Early production and Lost sale and will be denoted by ULS-TW-EL. The second problem will focus on producing the demand uppermost within its time windows and then by allowing the violation of the release or the due period. This leads to allowing respectively Early production or Backlog. The problem will be denoted by ULS-TW-EB. In this case, backlogs that occur in the very last period of the horizon can be considered as lost sales.

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2.1 A formulation for the problem with early production and lost sale

The following model considers the uncapacitated non customer specific case with lost sale and early production (ULS-TW-EL). Let X_t be the quantity processed in period t, I_t the inventory level at the end of period t and Y_t the setup binary variable which is equal to 1 if $X_t > 0$ and 0 otherwise. The lost sale at period t is defined by L_t and the quantity started at period t *or before* and that is not available at t is represented by E_t.

In order to generalize the classical demand time window constraints for the ULS-TW-EL, we need to disaggregate the lost sale variables. The production time window constraints can be written as follows: $\sum_{k=1}^{t} X_k \leq \sum_{k=1}^{t} \sum_{l=k}^{T} d_{kl}$, $\forall\ t = 1, \cdots, T$. If we consider only lost sale L_l corresponding to periods $l \in [1, \ldots, t]$ and the early delivery E_t at period t, then the time window constraints are not sufficient. In fact, lost sale related to a demand d_{kl} that begins before or at period t and ends after period t must be taken into account. We define the new disaggregate lost sale variable L_{lt} that represents the lost sales at period t for a demand that starts at period l, such that $L_t = \sum_{l=1}^{t} L_{lt}$.

$$\text{Minimize} \sum_{t=1}^{T} \left(p_t X_t + s_t Y_t + h_t I_t + l_t \sum_{k=1}^{t} L_{kt} + e_t E_t \right) \tag{1}$$

Subject to:

$$I_{t-1} + X_t + \sum_{k=1}^{t} L_{kt} = D_t + I_t \qquad\qquad \forall\ t = 1, \cdots, T \tag{2}$$

$$\sum_{k=1}^{t} \left(X_k + \sum_{l=k}^{T} L_{kl} \right) - E_t \leq \sum_{k=1}^{t} \sum_{l=k}^{T} d_{kl} \qquad\qquad \forall\ t = 1, \cdots, T \tag{3}$$

$$X_t \leq \left(\sum_{l=1}^{T} D_l \right) Y_t \qquad\qquad \forall\ t = 1, \cdots, T \tag{4}$$

$$L_{kt} \leq d_{kt} \qquad\qquad \forall\ k, t = 1, \cdots, T\ (k \leq t) \tag{5}$$

$$Y_t \in \{0, 1\} \qquad\qquad \forall\ t = 1, \cdots, T \tag{6}$$

$$L_{kt} \geq 0 \qquad\qquad \forall\ k, t = 1, \cdots, T\ (k \leq t) \tag{7}$$

$$X_t,\ E_t,\ I_t \geq 0 \qquad\qquad \forall\ t = 1, \cdots, T \tag{8}$$

The objective (1) is to minimize the total production, setup, lost sale, early production and holding costs. Constraints (2) are the inventory balance equations. In Constraints (3), the cumulative quantity of production and lost sales for the first t periods should be at most equal to the sum of the demands that must start at or before t and delivered at or after t plus the early production at period t. Constraints (4) relate the binary setup variables to the continuous production variables. Constraints (5) define upper bounds on the lost sale variables. The domain definitions of the variables are defined in Constraints (6)-(8).

2.2 Illustrative example

The following example illustrates an optimal solution structure for the ULS-TW-EL problem. The parameters of the instance are given in Table 1 and in Figure 1.

Periods	1	2	3	4	5
s_t	500	5000	5000	5000	500
h_t	4	4	4	4	4
l_t	100	100	70	30	30
e_t	20	40	20	10	10
p_t	2	2	2	2	2

Table 1: Example with early production and lost sale

The optimal solution of the problem is obtained using the mathematical model of Section 2.1 and a Mixed Integer Linear Programming (MILP) solver. The solution is depicted in Figure 1.
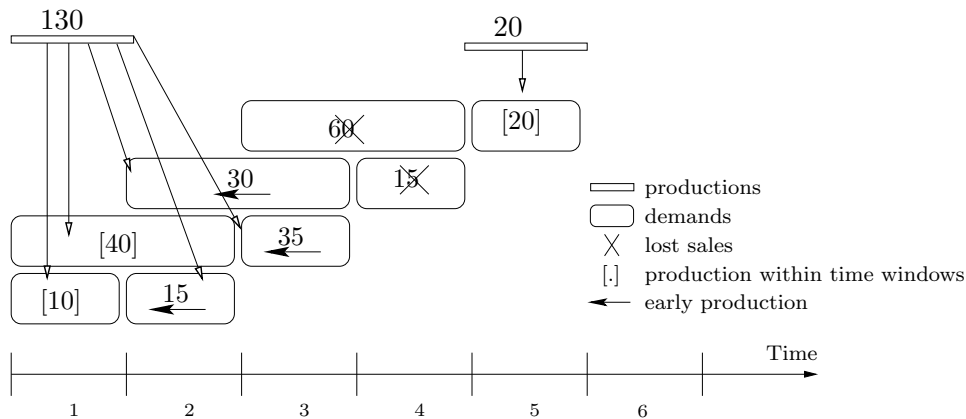

Figure 1: Early production and lost sale

From Figure 1, it is interesting to notice that, due to the cost values, demands d_{22}, d_{23} and d_{33} are early produced in period 1 while demands d_{11}, d_{12} and d_{55} are produced during their time windows. We can also observe that demands d_{34} and d_{44} are totally lost.

2.3 A formulation of the problem with early production and backlog

Let B_t be the quantity backlogged at the end of period t. The following model considers the non customer specific case with backlog and early production denoted by ULS-TW-EB.

$$\text{Minimize} \sum_{t=1}^{T} \left(p_t X_t + s_t Y_t + h_t I_t + b_t B_t + e_t E_t \right) \tag{9}$$

Subject to:

$$I_{t-1} + X_t + B_t = D_t + I_t + B_{t-1} \qquad\qquad \forall \ t = 1, \cdots, T \tag{10}$$

$$\sum_{k=1}^{t} X_k - E_t + B_t \leq \sum_{k=1}^{t} \sum_{l=k}^{T} d_{kl} \qquad\qquad \forall \ t = 1, \cdots, T \tag{11}$$

$$X_t \leq \left(\sum_{l=1}^{T} D_l \right) Y_t \qquad\qquad \forall \ t = 1, \cdots, T \tag{12}$$

$$Y_t \in \{0, 1\} \qquad\qquad \forall \ t = 1, \cdots, T \tag{13}$$

$$X_t, \ B_t, \ E_t, \ I_t \geq 0 \qquad\qquad \forall \ t = 1, \cdots, T \tag{14}$$

The objective (9) is to minimize the total production, setup, early production, backlog and holding costs. Constraints (10) are the inventory balance equations. In Constraints (11), the cumulative quantity of production for the first t periods should be at most equal to all the demands that must start at or before t and delivered at or after t plus the early production minus the backlogged quantities at period t. Constraints (12), (13) and (14) are similar to Constraints (4), (6) and (8) defined in Section 2.1.

2.4 Illustrative example

The following example illustrates an optimal solution structure for the ULS-TW-EB problem. The parameters are given in Table 2 and in Figure 2.

Periods	1	2	3	4	5
s_t	500	5000	5000	5000	500
h_t	4	4	4	4	4
b_t	30	30	30	30	30
e_t	20	40	20	10	10
p_t	2	2	2	2	2

Table 2: Example with early production and backlog

The optimal solution of the problem is obtained using the mathematical model of Section 2.3 and a Mixed Integer Linear Programming (MILP) solver. The solution is depicted in Figure 2.

From Figure 2, it is interesting to notice that demands d_{22} and d_{23} are early produced in period 1 while demands d_{11} and d_{12} are produced within their time windows. We can also observe that demands d_{33}, d_{34} and d_{44} are totally backlogged.
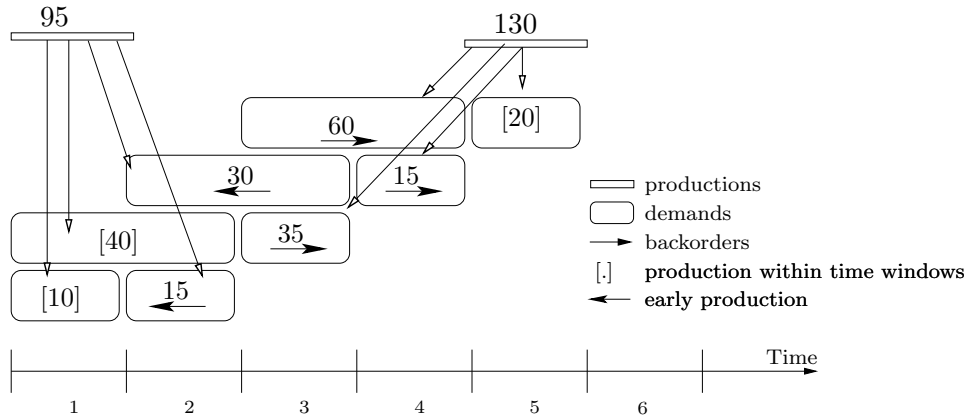
Figure 2: Early production and backlog

2.5 A formulation for the problem with early production, backlog and lost sale

To formulate the model that considers the non customer specific case with Early production, Lost sale and Backlog denoted by ULS-TW-ELB, we can merge the two previous models. The new model is constructed by modifying the one described through Constraints (9)-(14). The objective function (9) is modified by introducing the cost related to lost sales. We need to introduce lost sale variables in Constraints (10) and (12) as Constraints (2) and (3), and to add Constraints (5) and (7).

As previously mentioned, these three options (early production, backlog and lost sale) are generally not considered simultaneously. For some cases (e.g. $l_t \gg b_t > p_t$), we can switch from the ULS-TW-ELB model to the ULS-TW-EB model by considering the demands that are backlogged to the end of the horizon as lost sales (with $b_T \geq 0$). We can also define two levels of time windows for each demand. The first one is equivalent to the classical one (let us call it *weak time window*), it can be violated by considering early production and backlog. The second one is calculated according to lost sale, backlog and early production costs (let us call it *strong time window*), no backlog or early production is allowed for this kind of time windows. A preprocessing phase can be used to define the weak and strong time windows that can improve the resolution of the problem.

3 Structural optimality properties

This section presents several structural characteristics of optimal solutions of the uncapacitated lot-sizing problem with production time windows under the non customer specific (NCS) assumption. The objective is to minimize the total cost including production, setup and holding costs as well as early production, backlog and lost sale costs.

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3.1 Properties based on time windows

We first recall the non-inclusive time window property for the NCS case [7].

Property 1 (Non inclusive time windows). *When the time windows are non customer specific, then $d_{st} > 0$ implies that $d_{kl} = 0$, for all $k < s$ and for all $l > t$. Thus, two time windows cannot be strictly included, i.e for any pair of time windows $((s_1, t_1), (s_2, t_2))$, either $s_1 \leq s_2$ and $t_1 \leq t_2$, or $s_1 \geq s_2$ and $t_1 \geq t_2$.*

If we consider the release (resp. delivery) vector defined by the release (resp. delivery) period r (resp. t) of each demand d_{rt}, we define a unique set of non inclusive time windows associated with these vectors. Let us denote by K the cardinal of this set. The non inclusive time windows property of the NCS problem gives an upper bound on K, that is $2T - 1$. In the sequel of the paper, the unique set of non-inclusive time windows will be denoted by d^k for $k = 1, \cdots, K$. The reader can refer to [5] for a complete description of the algorithm that converts the time windows to NCS demands.

An important property satisfied by at least one optimal solution of the ULS-TW-EL problem is that a demand d_{st} is either fully produced (possibly before period s) or fully lost.

Theorem 1. *There always exists an optimal solution of the ULS-TW-EL problem in which the demand d_{st} is processed in only one period k such that $0 \leq k \leq t$ or entirely lost at t.*
    In the following, the hat-superscripted variables such as $\widehat{x}_t$ refer to optimal values.

Proof. The proof is made by contradiction. Let us assume that in an optimal solution the production of a demand d_{st} takes place in periods t_1 and t_2 such that $t_1 < s \leq t_2 \leq t$. We will denote $\widehat{x}_{t_1}$ (resp. $\widehat{x}_{t_2}$) the production of period t_1 (resp. t_2). The variable cost of producing the quantity $\widehat{x}_{t_1}$ at period t_1 is equal to $C_{t_1} = \sum_{k=t_1}^{s-1} e_t \widehat{x}_{t_1} + \sum_{k=t_1}^{t-1} h_k \widehat{x}_{t_1} + p_{t_1} \widehat{x}_{t_1}$, and the variable cost of producing the quantity $\widehat{x}_{t_2}$ at period t_2 is equal to $C_{t_2} = \sum_{k=t_2}^{t-1} h_k \widehat{x}_{t_2} + p_{t_2} \widehat{x}_{t_2}$. If $C_{t_1} < C_{t_2}$ then, by decreasing the production at period t_2 by $\widehat{x}_{t_2}$ and increasing the production at period t_1 by $\widehat{x}_{t_2}$, we obtain a solution that dominates the previous one since the total cost is reduced by $(C_{t_2} - C_{t_1})$. Similarly, if $C_{t_1} > C_{t_2}$ then, by decreasing the production at period t_1 by $\widehat{x}_{t_1}$ and increasing the production at period t_2 by $\widehat{x}_{t_1}$, we obtain a solution that dominates the previous one. When $C_{t_1} = C_{t_2}$, a solution where the production of d_{st} occurs in a single period (t_1 or t_2) can be derived without increasing the total cost. This proves that the initial solution could not be a dominant solution, which is a contradiction. The proof uses the same arguments on the lost sale and production variables for the case where a demand d_{st} is partially lost. ☐

Similarly, another important property satisfied by at least one optimal solution of the ULS-TW-EB problem is that a demand d_{st}, is either fully produced (possibly

before period s) or fully backlogged.

Theorem 2. *There is always an optimal solution of the ULS-TW-EB problem in which the demand d_{st} is processed in only one period k such that $0 \leq k \leq t$ or backlogged in only one period k such that $t < k \leq T$.*

Proof. The proof uses the same arguments as the ones given in the proof of Theorem 1 on the backlog and production variables for the case where a demand d_{st} is partially backlogged. $\square$

We derive the following corollaries from Theorems 1 and 2 where $[r_k, t_k]$ is the time window associated to d^k.

Corollary 1. *If d^k and d^{k+1} are processed in an optimal solution of the ULS-TW-EL (or ULS-TW-EB) problem, then d^k is processed before or at the same period as d^{k+1}.*

Proof. The proof is similar to the one presented in Dauzère-Pérès *et al.* [7]. $\square$

Particularly, we can observe that if the production of both demands d^k and d^{k+1} occur in the interval $[r_{k+1}, t_k]$, then the demands are processed at the same period for the ULS-TW-EL (or ULS-TW-EB) problem.

Corollary 2. *In an optimal solution of the ULS-TW-EL (or ULS-TW-EB) problem, if d^k is early processed then d^l such that $k < l \leq K$ and $r_k = r_l$ cannot be processed in $[r_k, t_k]$.*

Proof. The proof is made by contradiction. Suppose that in an optimal solution, a demand d^k is early produced at period $t_1 < r_k$ and a demand d^l (with $k < l \leq K$ and $r_l = r_k$) is processed at period t_2 such that $r_k \leq t_2 \leq t_l$. $\hat{x}_{t_1}^k$ (resp. $\hat{x}_{t_2}^l$) denotes the quantity produced at period t_1 (resp. t_2) to satisfy the demand d^k (resp. d^l). Let us define $C_{t_1,t_2} = \sum_{t=t_1}^{r_k-1} e_t + \sum_{t=t_1}^{t_2-1} h_t + p_{t_1}$. The solution is not dominant since it can be improved, by moving the production $\hat{x}_{t_2}^l$ to period t_1 if $C_{t_1,t_2} < p_{t_2}$ or by moving the production $\hat{x}_{t_1}^k$ to period t_2 if $C_{t_1,t_2} > p_{t_2}$. When $C_{t_1,t_2} = p_{t_2}$, a solution where the production of d^k occurs in a single period (t_1 or t_2) can be derived without increasing the total cost. This proves that the initial solution could not be a dominant solution, which is a contradiction. $\square$

Corollary 3. *In an optimal solution of the ULS-TW-EL problem, if customer order d^k is lost then d^l is also lost for all l such that $k < l \leq K$ and $t_l = t_k$.*

Proof. The proof is made by contradiction. Suppose that, in an optimal solution, a demand d^k is totally lost and a demand d^l (with $k < l \leq K$ and $t_l = t_k$) is processed at period t such that $t \leq t_l$. Let us denote C_t^k the unitary cost of satisfying the demand d^k from period t, $C_t^k = \sum_{t'=t}^{r_k-1} e_{t'} + \sum_{t'=t}^{t_k-1} h_{t'} + p_t$. It is obvious that $C_t^k \leq C_t^l$. This solution is not a dominant one since it can be improved, by producing the demand d^k at period t if $C_{t,k} \leq C_{t,l} \leq l_{t_k}$ or by allowing the demand d^l to be lost if $C_t^l > l_{t_k}$. This proves that the initial solution could not be a dominant solution, which is a contradiction. $\square$

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Note that we cannot derive a similar property when $t_l > t_k$. Indeed, in an optimal solution of the ULS-TW-EL problem, we can observe the following situation: d^k is lost while d^l such that $k < l \leq K$ and $t_l > t_k$ is processed at period j with $r_k \leq j \leq t_k$. In this case, it is less costly to produce d_l at period j rather than assuming a lost sale at period d_l contrary to demand d_k where the lost sale at period d_k is less costly than producing before d_k.

Corollary 4. *In an optimal solution of the ULS-TW-EB problem where d^k is backlogged then all the demands d^l such that $k < l \leq K$ and $t_l = t_k$ are backlogged.*

Proof. The proof is made by contradiction. Suppose that in an optimal solution, a demand d^k is totally backlogged at period t_2 such that $t_2 > t_k$ and a demand d^l (with $k < l \leq K$ and $t_l = t_k$) is processed at period t_1 such that $t_1 \leq t_l$. Let us denote by $C_{t_2}^k$ the unitary cost of satisfying the demand d^k by producing at period t_2, we have $C_{t_2}^k = \sum_{t'=r_k}^{t_2-1} b_{t'} + p_{t_2}$. Similarly, let us denote by $C_{t_1}^l$ the unitary cost of satisfying the demand d^l by a production at period t_1, then $C_{t_1}^l = \sum_{t'=t_1}^{r_k-1} e_{t'} + \sum_{t'=t_1}^{t_k-1} h_{t'} + p_{t_1}$. It is clear that $C_{t_1}^k \leq C_{t_1}^l$. This solution is not a dominant one since it can be improved, by producing the demand d^k at period t_1 if $C_{t_1}^k \leq C_{t_1}^l < C_{t_2}^k$ or by backlogging the demand d^l at period t_2 if $C_{t_1}^l > C_{t_2}^k$. When $C_{t_1}^l = C_{t_2}^k$, a solution where the production of d^l is backlogged to period t_2 or d^k is produced at period t_2 can be derived without increasing the total cost. This proves that the initial solution could not be a dominant solution, which is a contradiction. $\square$

3.2 Extended Wagner-Within properties

The following lemmas show that the well-known Wagner-Within based properties [13] are not observed when considering early production, backlog and lost sale. Therefore, we prove that some extensions are valid.

Lemma 1. *The zero-inventory ordering policy of the optimal solution, $\widehat{I}_{t-1}\widehat{X}_t = 0$ $\forall\, t = 1, \ldots, T$, is not valid for the ULS-TW-EL and ULS-TW-EB problems.*

Proof. It is possible to find a counter-example where an optimal solution of an instance with two demands d^k and d^l such that $k < l$ and $t_k = t_l$ consists in producing at the release period of both demands We have $\widehat{X}_{r_k} > 0$ (to satisfy demand d^k) and $\widehat{X}_{r_l} > 0$ (to satisfy demand d^l), and thus $\widehat{I}_{r_l-1}\widehat{X}_{r_l} > 0$. $\square$

For the uncapacitated lot-sizing problem with lost sales (ULS-LS), $\widehat{L}_t\widehat{X}_t = 0$ (when $p_t < l_t$) and then $\widehat{L}_t(D_t - \widehat{L}_t) = 0$ (see [4]). When considering time windows, the property does not hold.

Lemma 2. $\widehat{L}_t(D_t - \widehat{L}_t) = 0$ $\forall\, t = 1, \ldots, T$, *is not valid for the ULS-TW-EL problem.*

Proof. Similarly to the proof of Lemma 1, it is possible to find a counter-example where an optimal solution of an instance with two demands d^k and d^l such that

$k < l$ and $t_k = t_l$ with d^k produced at period r_k and d^l totally lost (i.e. $\widehat{L}_{t_l} > 0$), so that $\widehat{L}_{t_k}(D_{t_k} - \widehat{L}_{t_k}) > 0$ since $D_{t_k} = d^k + d^l$. $\qquad\square$

The following property illustrates the fact that, in an optimal solution of the ULS-TW-EL problem, if a procurement is made at period t, then lost sales cannot be allowed in the same period. Let us recall that lost sale costs are larger than production costs as they act as penalties on the non-delivery of the demand requirement.

Lemma 3. $\widehat{L}_t\widehat{X}_t = 0$ when $p_t < l_t$ $\forall t = 1, \ldots, T$ for the ULS-TW-EL problem.

Proof. Let us assume that $X_t > 0$ and $L_t > 0$ at optimality. Then adding a value δ to X_t and reducing L_t by δ such that $\delta \le D_t - X_t$, will lower the total cost by $(p_t - l_t)\delta$. Thus, the solution is not optimal. $\qquad\square$

Lemma 3 is not valid for ULS-LS and ULS-TW-EL if the condition $p_t < l_t$ is not satisfied. In fact, it is possible to find a counter-example where the optimal solution of an instance with two demands d^k and d^l, such that $k < l$ and $t_k < t_l$, with a structure of production and lost sale costs allowing $X_{t_k} > 0$ to satisfy d^l and d^k is lost if $p_{t_k} > l_{t_k}$.

Note also that no early production can occur at period $t - 1$ if a production takes place at period t in an optimal solution of the ULS-TW-EL and ULS-TW-EB problems.

Lemma 4. $\widehat{E}_{t-1}\widehat{X}_t = 0$, $\forall t = 1, \ldots, T$ for the ULS-TW-EL and ULS-TW-EB problems.

Proof. The proof is made by contradiction. Suppose that in an optimal solution, a demand d^k is totally early produced at period t_1 such that $t_1 < r_k$ and a production is made at period t_2 for a given demand d^l (with $t_2 \ge t_1$). According to the assumptions, the unitary cost of early producing at period t_1 and holding the unit until period t_2 for demand d^k is smaller than the unitary cost of producing at period t_2, i.e.

$$C_{t_1,t_2}^k < p_{t_2} \tag{15}$$

with

$$C_{t_1,t_2}^k = \sum_{t'=t_1}^{t_2-1} e_{t'} + \sum_{t'=t_1}^{t_2-1} h_{t'} + p_{t_1} \tag{16}$$

On the other hand, the unitary cost of producing at period t_1 and holding the unit until t_2 for demand d^l is larger than the unitary cost of producing at period t_2, i.e.

$$C_{t_1,t_2}^l > p_{t_2} \tag{17}$$

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with

$$C_{t_1,t_2}^l = \sum_{t'=t_1}^{\min(r_l,t_2)-1} e_{t'} + \sum_{t'=t_1}^{t_2-1} h_{t'} + p_{t_1} \tag{18}$$

From (15) and (17) we have:

$$C_{t_1,t_2}^l > C_{t_1,t_2}^k \tag{19}$$

From (16) and (18) we have:

$$C_{t_1,t_2}^l \leq C_{t_1,t_2}^k \tag{20}$$

which contradicts (19). □

Recall that the Zangwill property [15] for the uncapacitated lot-sizing problem with backlogs still hold for the ULS-TW-EB problem, i.e. $\widehat{B}_t \widehat{X}_t = 0 \; \forall \, t = 1, \ldots, T$.

4   Dynamic programming algorithms

Without loss of generality, we consider that all the demands are ordered as described above from $k = 1, \ldots, K$, i.e. from the earliest to the latest.

Regarding the properties described in Section 3, we can derive dynamic programming algorithms to solve the ULS-TW-EL, ULS-TW-EB and ULS-TW-ULB problems. The rationale is mainly based on the following:

- The orders are examined from the earliest to the latest. If order k is produced, then it is produced before or at the same time as order $k + 1$ (see Corollary 1).

- If order k is produced (even early or late), then it is entirely produced in a single period (see Theorem 2).

- If order k is lost, then it is entirely lost at its due period (see Theorem 1).

In what follows, forward dynamic programming algorithms are proposed to compute the optimal values of ULS-TW-EL, ULS-TW-EB and ULS-TW-ELB problems based on recursion formula that illustrates the principles described above.

4.1   Solving the ULS-TW-EL problem

The dynamic programming algorithm uses the following new cost and parameter definitions.

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Definition 1. p'_{kt} denotes the cost of meeting a unitary demand in period t by a production in period k ($k \leq t$). The value of p'_{kt} is determined by (21).

$$p'_{kt} = p_k + \sum_{l=k}^{t-1} h_l, \ k \leq t \tag{21}$$

Definition 2. e'_{kt} denotes the cost of early producing a unitary demand that starts at period t from period k ($k \leq t$). The value of e'_{kt} is determined by (22).

$$e'_{kt} = \sum_{l=k}^{t} e_l \tag{22}$$

Definition 3. $H(t,k)$ is the value of an optimal solution for periods $1, \cdots, t$ in which demands $d^1, \cdots, d^k$ are either lost or produced before or at period t.

Definition 4. $G(t,k)$ is the value of an optimal solution for periods $1, \cdots, t$ in which demands $d^1, \cdots, d^{k-1}$ are either lost or produced before or at period t, a production is done at period t and d^k is either produced at period t or lost.

The optimal value of the objective function of problem ULS-TW-EL defined by (1)-(8) is given by $H(T,K)$. $H(T,K)$ is computed by a forward recursion starting from k $= 1$ with the initial conditions given by: $H(0,k) = \infty$ for $k > 0$, $H(t,0) = 0$ for $t > 0$, $H(0,0) = 0$ and $G(0,k) = \infty$ for $k > 0$, $G(t,0) = s_t$ for $t > 0$, $G(0,0) = 0$.

Algorithm 1 describes the dynamic programming algorithm which provides the optimal policy of procurements and losses over the time horizon. Some parameters are defined below to store at each step of the algorithm the variables used to compute $H(t,k)$ and $G(t,k)$. They mainly represent the last period and demand as well as an indicator to check whether a demand is produced or lost and the production period for each satisfied demand d^k. More precisely:

- $H^f(t,k)$ (respectively $G^f(t,k)$) represents the function used to calculate $H(t,k)$ (respectively $G(t,k)$). These values are fixed to G (respectively H) if the function used is G (respectively H).

- $H^t(t,k)$ (respectively $G^t(t,k)$) represents the last period of function H^f (respectively G^f).

- $H^k(t,k)$ (respectively $G^k(t,k)$) represents the last demand of the function H^f (respectively G^f).

- $H^p(t,k)$ and $G^p(t,k)$ are equal to -1 if demand d^k is lost or 1 of it is produced.

- J_k^* represents the production period of demand d^k. This value is equal to -1 if demand d^k is lost.

Algorithm 1 solving ULS-TW-EL

1: $G(0,0) \leftarrow 0$, $H(0,0) \leftarrow 0$
2: **for all** $k \in \{1, \ldots, K\}$ **do**
3:    $H(0,k) \leftarrow +\infty$, $G(0,k) \leftarrow +\infty$
4: **end for**
5: **for all** $t \in \{1, \ldots, T\}$ **do**
6:    $G(t,0) \leftarrow s_t$, $H(t,0) \leftarrow 0$
7: **end for**
8: **for** $t = 1$ to T **do**
9:    **for** $k = 1$ to K **do**
10:       **if** $(t \geq r_k)$ and $(t \leq t_k)$ **then**
11:          $G(t,k) \leftarrow \min(H(t-1,k-1) + s_t + p'_{tt_k}d^k, G(t,k-1) + p'_{tt_k}d^k, H(t-1,k-1) + s_t + l_{t_k}d^k, G(t,k-1) + l_{t_k})$
12:          Update $G^f(t,k)$, $G^t(t,k)$, $G^k(t,k)$ and $G^p(t,k)$.
13:       **end if**
14:       **if** $(t \leq r_k)$ **then**
15:          $G(t,k) \leftarrow \min(H(t-1,k-1) + s_t + (p'_{tt_k} + e'_{tr_k})d^k, G(t,k-1) + (p'_{tt_k} + e'_{tr_k})d^k, H(t-1,k-1) + s_t + l_{t_k}d^k, G(t,k-1) + l_{t_k}d^k)$
16:          Update $G^f(t,k)$, $G^t(t,k)$, $G^k(t,k)$ and $G^p(t,k)$.
17:       **end if**
18:       **if** $(t \geq t_k)$ **then**
19:          $G(t,k) \leftarrow +\infty$
20:       **end if**
21:       $H(t,k) \leftarrow \min(H(t-1,k), G(t,k), H(t-1,k-1) + l_{t_k}d^k, H(t,k-1) + l_{t_k}d^k)$
22:       Update $H^f(t,k)$, $H^t(t,k)$, $H^k(t,k)$ and $H^p(t,k)$.
23:    **end for**
24: **end for**
25: $k \leftarrow K$, $t \leftarrow T$, $F \leftarrow H$
26: **repeat**
27:    **if** $(F = H)$ **then**
28:       **if** $(H^p(t,k) = -1)$ **then**
29:          $J_k^* \leftarrow -1$
30:       **end if**
31:       $F \leftarrow H^f(t,k)$, $t \leftarrow H^t(t,k)$, $k \leftarrow H^k(t,k)$
32:    **else**
33:       **if** $(G^p(t,k) = -1)$ **then**
34:          $J_k^* \leftarrow -1$
35:       **else**
36:          $J_k^* \leftarrow t$
37:       **end if**
38:       $F \leftarrow G^f(t,k)$, $t \leftarrow G^t(t,k)$, $k \leftarrow G^k(t,k)$
39:    **end if**
40: **until** $(k = 0)$

Loops in Algorithm 1 contain at most two levels with respectively T and K iterations. Recall that, in the NCS problem, $K \leq 2T - 1$. The complexity of Algorithm 1 that solves ULS-TW-EL is thus $O(T^2)$. To obtain the optimal policy of procurements and losses for each demands d^k, we need to backtrack from the final $H(T, K)$ through all k using the $H^f(t, k)$, $H^t(t, k)$, $H^k(t, k)$, $H^p(t, k)$, $G^f(t, k)$, $G^t(t, k)$, $G^k(t, k)$ and $G^p(t, k)$ values. This procedure computes the J_k^* values that represent the production period of demand d^k or -1 if this demand is lost. The algorithm stops when a value J_k^* is assigned to each demand d^k.

Implementation of Algorithm 1 on Example 2.2

Algorithm 1 is applied to the example described in Table 1. Tables 3 and 4 give respectively the values of $G(t, k)$ and $H(t, k)$ followed by two parameters. The first parameter is equal to (-1) if the demand k is lost at period t, this parameter is given by $G^p(t, k)$ for $G(t, k)$ and $H^p(t, k)$ for $H(t, k)$. The second parameter represents the function used to calculate the current values of $G(t, k)$ and $H(t, k)$, this parameter is given by the values of $G^f(t, k)$, $G^t(t, k)$, $G^k(t, k)$ for $G(k, t)$ and the values of $H^f(t, k)$, $H^t(t, k)$, $H^k(t, k)$ for $H(t, k)$. The values in bold represent the steps of the backtrack procedure defined in Algorithm 1. Finally, the optimal policy of procurements and losses for each demands d^k is provided in Table 5.

	k	1	2	3	4	5	6	7	8
t	1	**520**	**760**	**1150**	**2050**	**4500**	**6300(-1)**	**6750(-1)**	7350(-1)
		G(1,0)	**G(1,1)**	**G(1,2)**	**G(1,3)**	**G(1,4)**	**G(1,5)**	**G(1,6)**	G(1,7)
	2	$+\infty$	5600	5630	5810	7420	9220(-1)	9670(-1)	10270(-1)
			H(1,1)	G(2,2)	G(2,3)	G(2,4)	G(2,5)	G(2,6)	G(2,7)
	3	$+\infty$	$+\infty$	$+\infty$	6210	6280	6640	7030	7630(-1)
					H(2,3)	G(3,4)	G(3,5)	G(3,6)	G(3,7)
	4	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	9620	9650	9970
							H(3,5)	G(4,6)	G(4,7)
	5	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	**7290**
									H(4,7)

Table 3: $G(t, k)$ values for Example 2.2

	k	1	2	3	4	5	6	7	8
t	1	520	760	1150	2050	4500	6300	**6750**	7350
		G(1,1)	G(1,2)	G(1,3)	G(1,4)	G(1,5)	G(1,6)	**G(1,7)**	G(1,8)
	2	520	760	1150	2050	4500	6300	**6750**	7350
		H(1,1)	H(1,2)	H(1,3)	H(1,4)	H(1,5)	H(1,6)	**H(1,7)**	H(1,8)
	3	520	760	1150	2050	4500	6300	**6750**	7350
		H(2,1)	H(2,2)	H(2,3)	H(2,4)	H(2,5)	H(2,6)	**H(2,7)**	H(2,8)
	4	520	760	1150	2050	4500	6300	**6750**	7350
		H(3,1)	H(3,2)	H(3,3)	H(3,4)	H(3,5)	H(3,6)	**H(3,7)**	H(3,8)
	5	520	760	1150	2050	4500	6300	6750	**7290**
		H(4,1)	H(4,2)	H(4,3)	H(4,4)	H(4,5)	H(4,6)	H(4,7)	**G(5,8)**

Table 4: $H(t, k)$ values for Example 2.2

k	1	2	3	4	5	6	7	8
J_k^*	1	1	1	1	1	-1	-1	5

Table 5: The optimal policy of procurements and losses for Example 2.2

4.2 Solving the ULS-TW-EB problem

The dynamic programming algorithm uses the following new cost and parameter definitions.

Definition 5. b'_{kt} *denotes the cost of meeting a unitary demand in period* t *by a production in period* k *($k > t$). The value of* b'_{kt} *is determined by (23).*

$$b'_{kt} = p_k + \sum_{l=t}^{k-1} b_l, \ k > t \tag{23}$$

Definition 6. $H(t, k)$ *is the value of an optimal solution for periods* $1, \cdots, t$ *in which demands* $d^1, \cdots, d^k$ *are produced before or at period* t.

Definition 7. $G(t, k)$ *is the value of an optimal solution for periods* $1, \cdots, t$ *in which demands* $d^1, \cdots, d^{k-1}$ *are either produced before or at period* t *and* d^k *is produced at period* t. *When* $t \geq t_k$, $G(t, k)$ *is computed using the cumulative backlogging costs.*

$H(T, K)$ represents the optimal value of the objective function of problem ULS-TW-EB defined by (9)-(14). $H(T, K)$ is computed by a forward recursion starting from k = 1 with the initial conditions given by: $H(0, k) = \infty$ for $k > 0$, $H(t, 0) = 0$ for $t > 0$, $H(0, 0) = 0$ and $G(0, k) = \infty$ for $k > 0$, $G(t, 0) = s_t$ for $t > 0$, $G(0, 0) = 0$.

The following parameters defined in Section 4.1 will also be used:

- p'_{kt} defined by Equation (21).

- e'_{kt} defined by Equation (22).

- J_k^*, $H^f(t, k)$, $H^t(t, k)$, $H^k(t, k)$, $G^f(t, k)$, $G^t(t, k)$ and $G^k(t, k)$.

In what follows, we present a forward dynamic programming algorithm to compute $G(t, k)$ and $H(t, k)$ and to determine whether a demand is early produced, backlogged or produced within its time window. In Algorithm 2, when $t \geq t_k$ the value of $G(t, k)$ is computed using the cumulative backlogging costs unlike Algorithm 1 where $G(t, k) = +\infty$ when $t \geq t_k$.

Loops in Algorithm 2 contain at most two levels with respectively T and K iterations. The complexity of Algorithm 2 that solves ULS-TW-EB is thus $O(T^2)$. To obtain the optimal policy of procurements and losses for each demands d^k, we need to backtrack from the final $H(T, K)$ through all k using the $H^f(t, k)$, $H^t(t, k)$,

Algorithm 2 solving ULS-TW-EB

1: $G(0,0) \leftarrow 0$, $H(0,0) \leftarrow 0$
2: **for all** $k \in \{1, \ldots, K\}$ **do**
3:    $H(0,k) \leftarrow +\infty$, $G(0,k) \leftarrow +\infty$
4: **end for**
5: **for all** $t \in \{1, \ldots, T\}$ **do**
6:    $G(t,0) \leftarrow s_t$, $H(t,0) \leftarrow 0$
7: **end for**
8: **for** $t = 1$ to T **do**
9:    **for** $k = 1$ to K **do**
10:       **if** $(t \geq r_k)$ and $(t \leq t_k)$ **then**
11:          $G(t,k) \leftarrow \min(H(t-1,k-1) + s_t + p'_{tt_k}d^k, G(t,k-1) + p'_{tt_k}d^k)$
12:          Update $G^f(t,k)$, $G^t(t,k)$ and $G^k(t,k)$.
13:       **end if**
14:       **if** $(t \leq r_k)$ **then**
15:          $G(t,k) \leftarrow \min(H(t-1,k-1) + s_t + (p'_{tt_k} + e'_{tr_k})d^k, G(t,k-1) + (p'_{tt_k} + e'_{tr_k})d^k)$
16:          Update $G^f(t,k)$, $G^t(t,k)$ and $G^k(t,k)$.
17:       **end if**
18:       **if** $(t \geq t_k)$ **then**
19:          $G(t,k) \leftarrow \min(H(t-1,k-1) + s_t + b'_{tt_k}d^k, G(t,k-1) + b'_{tt_k}d^k)$
20:          Update $G^f(t,k)$, $G^t(t,k)$ and $G^k(t,k)$.
21:       **end if**
22:       $H(t,k) \leftarrow \min(H(t-1,k), G(t,k))$
23:       Update $H^f(t,k)$, $H^t(t,k)$ and $H^k(t,k)$.
24:    **end for**
25: **end for**
26: $k \leftarrow K$, $t \leftarrow T$, $F \leftarrow H$
27: **repeat**
28:    **if** $(F = H)$ **then**
29:       $F \leftarrow H^f(t,k)$, $t \leftarrow H^t(t,k)$, $k \leftarrow H^k(t,k)$
30:    **else**
31:       $J^*_k \leftarrow t$
32:       $F \leftarrow G^f(t,k)$, $t \leftarrow G^t(t,k)$, $k \leftarrow G^k(t,k)$
33:    **end if**
34: **until** $(k = 0)$

$H^k(t, k)$, $G^f(t, k)$, $G^t(t, k)$ and $G^k(t, k)$ values. This procedure obtains the J_k^* values that represents the production period of demand d^k. It stops when a value J_k^* is assigned to each demand d^k.

Implementation of Algorithm 2 on Example 2.4

Algorithm 2 is applied to the example described in Table 2. Tables 6 and 7 give respectively the values of $G(t, k)$ and $H(t, k)$ followed by one parameters. This parameter represents the function used to calculate the current values of $G(t, k)$ and $H(t, k)$, it is given by the values of $G^f(t, k)$, $G^t(t, k)$, $G^k(t, k)$ for $G(k, t)$ and the values of $H^f(t, k)$, $H^t(t, k)$, $H^k(t, k)$ for $H(t, k)$. The values in bold represent the steps of the backtrack procedure defined in Algorithm 2. Finally, the optimal policy of procurements for each demands d^k is provided in Table 8.

	k	1	2	3	4	5	6	7	8
t	1	**520**	**760**	**1150**	**2050**	4500	8940	10350	12510
		G(1,0)	**G(1,1)**	**G(1,2)**	**G(1,3)**	G(1,4)	G(1,5)	G(1,6)	G(1,7)
	2	5320	5400	5430	5610	7220	10220	11270	12950
		G(2,0)	G(2,1)	G(2,2)	G(2,3)	G(2,4)	G(2,5)	G(2,6)	G(2,7)
	3	5620	6800	6240	6210	6280	6640	7030	7830
		G(3,0)	H(2,1)	H(2,2)	H(2,3)	G(3,4)	G(3,5)	G(3,6)	G(3,7)
	4	5920	8000	6690	7110	8170	8290	8320	8640
		G(4,0)	H(3,1)	H(3,2)	H(3,3)	H(3,4)	G(4,5)	G(4,6)	G(4,7)
	5	1720	4700	2640	3510	**4720**	**6640**	**7120**	**7160**
		G(5,0)	H(4,1)	H(4,2)	H(4,3)	**H(4,4)**	**G(5,5)**	**G(5,6)**	**G(5,7)**

Table 6: $G(t, k)$ values for Example 2.4

	k	1	2	3	4	5	6	7	8
t	1	520	760	1150	**2050**	4500	8940	10350	12510
		G(1,1)	G(1,2)	G(1,3)	**G(1,4)**	G(1,5)	G(1,6)	G(1,7)	G(1,8)
	2	520	760	1150	**2050**	4500	8940	10350	12510
		H(1,1)	H(1,2)	H(1,3)	**H(1,4)**	H(1,5)	H(1,6)	H(1,7)	H(1,8)
	3	520	760	1150	**2050**	4500	6640	7030	7830
		H(2,1)	H(2,2)	H(2,3)	**H(2,4)**	H(2,5)	G(3,6)	G(3,7)	G(3,8)
	4	520	760	1150	**2050**	4500	6640	7030	7830
		H(3,1)	H(3,2)	H(3,3)	**H(3,4)**	H(3,5)	H(3,6)	H(3,7)	H(3,8)
	5	520	760	1150	2050	4500	6640	7030	**7160**
		H(4,1)	H(4,2)	H(4,3)	H(4,4)	H(4,5)	G(5,6)	H(4,7)	**G(5,8)**

Table 7: $H(t, k)$ values for Example 2.4

k	1	2	3	4	5	6	7	8
J_k^*	1	1	1	1	5	5	5	5

Table 8: The optimal policy of procurements for Example 2.4

4.3 Solving the ULS-TW-ELB problem

The dynamic programming algorithm uses the parameters defined in Sections 4.1 and 4.2. We only need to redefine the computation of the recursion functions $H(t,k)$ and $G(t,k)$ by replacing the two level loops of Algorithm 1 defined from line 8 to line 24 by the two loops below:

1: **for** $t = 1$ to T **do**
2:    **for** $k = 1$ to K **do**
3:       **if** $(t \geq r_k)$ and $(t \leq t_k)$ **then**
4:          $G(t,k) \leftarrow \min(H(t-1,k-1) + s_t + p'_{tt_k}d^k, G(t,k-1) + p'_{tt_k}d^k, H(t-1,k-1) + s_t + l_{t_k}d^k, G(t,k-1) + l_{t_k})$
5:          Update $G^f(t,k)$, $G^t(t,k)$, $G^k(t,k)$ and $G^p(t,k)$.
6:       **end if**
7:       **if** $(t \leq r_k)$ **then**
8:          $G(t,k) \leftarrow \min(H(t-1,k-1) + s_t + (p'_{tt_k} + e'_{tr_k})d^k, G(t,k-1) + (p'_{tt_k} + e'_{tr_k})d^k, H(t-1,k-1) + s_t + l_{t_k}d^k, G(t,k-1) + l_{t_k}d^k)$
9:          Update $G^f(t,k)$, $G^t(t,k)$, $G^k(t,k)$ and $G^p(t,k)$.
10:      **end if**
11:      **if** $(t \geq t_k)$ **then**
12:         $G(t,k) \leftarrow \min(H(t-1,k-1) + s_t + b'_{tt_k}d^k, G(t,k-1) + b'_{tt_k}d^k, H(t-1,k-1) + s_t + l_{t_k}d^k, G(t,k-1) + l_{t_k}d^k)$
13:         Update $G^f(t,k)$, $G^t(t,k)$, $G^k(t,k)$ and $G^p(t,k)$.
14:      **end if**
15:      $H(t,k) \leftarrow \min(H(t-1,k), G(t,k), H(t-1,k-1) + l_{t_k}d^k, H(t,k-1) + l_{t_k}d^k)$
16:      Update $H^f(t,k)$, $H^t(t,k)$, $H^k(t,k)$ and $H^p(t,k)$.
17:    **end for**
18: **end for**

The time complexity of the Algorithm is $O(T^2)$. Let us recall that solving the ULS-TW-ELB problem can also be equivalent to solving the ULS-TW-EB problem if the backlogging to the end of the horizon is allowed as well as early producing at period 0. In both cases, the backlogged quantities at period T or the early production at period 0 are considered as lost sales. However, if a maximal number of backlogging (or early producing) periods is imposed, then solving the ULS-TW-ELB problem is relevant. Finally, a preprocessing phase on the cost computation can be performed to define for each demand the strong and weak time windows as proposed in Section 2.5.

5 Conclusion

New practical variants of the single item uncapacitated lot-sizing problem with production time windows are proposed in this paper, when early production, lost sale and backlog are allowed. In our extended models, the objective is to minimize a function that aggregates production costs (fixed and variable costs), inventory costs, lost sale costs, early production costs and backlog costs. We proposed several properties of optimal solutions and derived dynamic programming algorithms with

an $O(T^2)$ time complexity where T denotes the number of periods. We showed that the zero-inventory ordering policy (Wagner and Whitin property) does not hold for lot-sizing problems with time windows. This property is replaced by a new one in which no early production can occur at period $t - 1$ if a production takes place at period t in an optimal solution. We also showed some other important properties on lost sale, early production and backlog variables as well as time windows that helped us in deriving the polynomial algorithms. The proposed algorithms can be used in a Lagrangian relaxation procedure to calculate a lower bound for the multi-item capacitated version of the proposed models. To obtain a near optimal solution for these multi-item capacitated lot-sizing problems, a Lagrangian heuristic can be developed to construct a feasible solution from the Lagrangian solutions. A further research direction can be devoted to derive some properties for the customer specific case with early production, lost sale and backlog.

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